Math 246A Lecture 25 Notes

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1 Equiboundedness of Normal Families and the Riemann Mapping Theorem

1.1 Equiboundedness of normal families

Last time, we were probing the following theorem about normal families.

Theorem 1.1. *Let $\mathcal{F} \subseteq H(\Omega)$. Then $\mathcal{F}$ is normal if and only if for all compact $K \subseteq \Omega$, there exists $M_K < \infty$ such that $\sup_{f \in \mathcal{F}} \sup_{z \in K} |f(z)| \leq M_K$.*

Proof. One half is he Arzelà-Ascoli theorem. Assume the latter condition. We can write $K \subseteq \bigcup_{j=1}^n B(z_j, \delta_j)$, where $\overline{B(z_j, 3\delta_j)} \subseteq \Omega$. Then let $\delta_0 = \min\{\delta j : 1 \leq j \leq n\} > 0$. Then $z \in B(z_j, 2\delta_j)$ implies that

$$f'(z) = \frac{1}{2\pi i} \int_{\partial B(z_j, 3\delta_j)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta,$$

for $f \in \mathcal{F}$, so

$$\sup_{z \in B(z_j, 2\delta_j)} |f'(z)| \leq 3M_K/\delta_K \leq 3M/\delta_0.$$

Now let $z, w \in K$ such that $z \in B(z_1, \delta_j)$ and $|w - z| < \delta_0 < \delta_j$. Then $[z, w] \subseteq B(z_j, 2\delta_j)$, so $|f(z) = f(w)| \leq 3M|z - w|/\delta_0$. So $\mathcal{F}$ is equicontinuous. $\square$

Remark 1.1. If f satisfies this condition, then f' does, as well.

1.2 The Riemann mapping theorem

Theorem 1.2 (Riemann mapping theorem)**.** *Let $\Omega \subseteq \mathbb{C}^*$ be a simply connected comain with $\#(\mathbb{C}^* \setminus \Omega) \geq 2$. Let $z_0 \in \Omega$. Then there exists a holomorphic $\varphi : \Omega \to \mathbb{D}$ that is 1 to 1, onto, and $\varphi(z_0) = 0$. Moreover φ is uniquely determined by*

$$\varphi'(z_0) = \sup\{|\psi'(z_0)| \mid \psi : \Omega \to \mathbb{D} \textit{ is holomorphic}, \psi(z_0) = 0\}.$$

Example 1.1. Suppose $\Omega = \mathbb{D}$ and $z_0 \neq 0$. An example of such a map is

$$Tz = \frac{z - z_0}{1 - \overline{z}_0 z}$$

We saw earlier that

$$T'(z_0) = \frac{1}{1 - |z_0|^2},$$

and Pick's theorem gives us that $|\psi'(z_0) \leq 1/(1 - |z_0|^2)$ with equality iff $\psi = e^{i\alpha}T$.

Proof. Let $\mathcal{F} = \{g : \Omega \to \mathbb{D} \mid g \text{ is holomorphic}, g \text{ is 1-1 on } \Omega, g(z_0) = 0, z'(z_0) > 0\}$. We first show that $\mathcal{F} \neq \varnothing$. Take $a \in \mathbb{C} \setminus \Omega$, and let $h(z) = \sqrt{z - a}$ for $z \in \Omega$; h exists because Ω is simply connected. The function h is 1-1 on Ω because if $\sqrt{z_1 - a} = \sqrt{z_2 - a}$, then $z_1 - a = z_2 - a$, so $z_1 = z_2$.To show that $h(z_0) \neq 0$, note that $h(\Omega) \supseteq \overline{B(h(z_0), \delta)}$ for some δ. $h(\Omega \cap \overline{B(-h(z), \delta)}) = \varnothing$. So $h(z_1) = -\zeta$ and $h(z_2) = \zeta$ imply that $z_1 = z_2$. Then let

$$H = \frac{\delta}{h(z) = h(z_0)}.$$

Then $H : \Omega \to \mathbb{D}$, H is 1-1, and $|H| < 1$. So

$$e^{i\alpha} \frac{H(z) - H(z_0)}{1 - \overline{H(z_0)} H(z)} = g \in \mathcal{F}.$$

Take $\{g_n\} \subseteq \mathcal{F}$ such that $g_n'(z_0) \to \sup_{\mathcal{F}} g'(z_0) < \infty$. Then there exist a subsequence $g - n_j$ that converges to g uniformly onall copact $K \subseteq \Omega$. Here, $g \in H(\Omega)$, and $|g| < 1$. Hurwitz's theorem gives us that g is 1-1 or g is constant. g is not constant, so $g \in \mathcal{F}$.

We now show that $g(\Omega) = \mathbb{D}$. Assume $\alpha \in \mathbb{D} \setminus g(\Omega)$. Let

$$G = \sqrt{\frac{g - \alpha}{1 - \overline{\alpha g}}}.$$

G exists because Ω is simply connected, G is 1-1, and $G(\Omega) \subseteq \mathbb{D}$. Now let

$$F = \frac{\overline{G'(z_0)}}{|G'(z_0)|} \frac{G - G(z_0)}{1 - \overline{G(z_0)} G}.$$

Then $G = \sqrt{T \circ g}$,where T is the Möbius transformation

$$Tw = \frac{w - \alpha}{1 - \overline{\alpha} w},$$

and $F = S \circ G$, where

$$S\zeta = \frac{\overline{G'(z_0)}}{|G'(z_0)|} \frac{\zeta - G(z_0)}{1 - \overline{G(z_0)} \zeta}.$$

So $g = T^{-1} \circ (S^{-1} \circ F)^2$. So, $g = A \circ F$, where $A9\zeta) = T^{-1}(S^{-1}(\zeta)^2)$. So $g'(z_0) = A'(0)F'(z_0)$, which means that $|A'(0)| \geq 1$ because $|F'(z_0)| \leq 1$ and g maximizes the derivative at 0. Then $A(\zeta) = e^{i\beta}\zeta$, but A is 2-1, so we have a contradiction. So there is no such α. $\square$